Isotropic realizability of a strain field for the two-dimensional incompressible elasticity system

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Isotropic realizability of a strain field for the incompressible two-dimensional Stokes equation

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Abstract

In the paper we study the problem of the isotropic realizability in $\mathbb{R}^2$ of a regular strain field $e(U) = \frac{1}{2}\left(DU + DU^T\right)$ for the incompressible Stokes equation, namely the existence of a positive viscosity $\mu > 0$ solving the Stokes equation in $\mathbb{R}^2$ with the prescribed field $e(U)$. We show that if $e(U)$ does not vanish at some point, then the isotropic realizability holds in the neighborhood of that point. The global realizability in $\mathbb{R}^2$ or in the torus is much more delicate, since it involves the global existence of a regular solution to a semilinear wave equation the coefficients of which depend on the derivatives of U. Using the semilinear wave equation we prove a small perturbation result: If DU is periodic and close enough to its average for the C^4-norm, then the strain field is isotropically realizable in a given disk centered at the origin. On the other hand, a counter-example shows that the global realizability in $\mathbb{R}^2$ may hold without the realizability in the torus, and it is discussed in connection with the associated semilinear wave equation. The case where the strain field vanishes is illustrated by an example. The singular case of a rank-one laminate field is also investigated.

Keywords: isotropic realizability, strain field, Stokes equation, first-order hyperbolic system, semilinear second-order hyperbolic equation

Mathematics Subject Classification: 35L05, 35L40, 35L71, 35Q30

1 Introduction

In the theory of composites (see, *e.g.*, [11]) the effective properties of a composite are classically obtained by the interactions of several isotropic phases periodically arranged, involving some periodic electric fields and current fields. It turns out that the related electric field may satisfies some constraints. Indeed, in two-dimensional conductivity Alessandrini and Nesi [2] showed the positivity of the determinant of the periodic matrix-valued electric field (each row of which corresponds to the vector electric field associated with one direction of the applied field). Hence, a two-dimensional matrix gradient field with a non-positive determinant cannot be an electric field. Therefore, it is natural to characterize the electric fields among all the possible gradient fields, and the current fields among all the possible divergence free fields. In this spirit a periodic gradient field ∇u is said to be isotropically realizable as an electric field in $\mathbb{R}^d$ if there exists a positive conductivity σ solving the equation

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

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The isotropic realizability holds in the torus if moreover the conductivity can be chosen periodic. Following [5] it is easy to build a periodic regular gradient field which is isotropically realizable in the whole space but not in the torus. In [5] we have completely characterized the set of the periodic regular gradients as isotropically realizable electric fields using a gradient flow approach. So, in dimension two a periodic regular gradient field is shown to be isotropically realizable in $\mathbb{R}^2$, if and only if it does not vanish in $\mathbb{R}^2$. Moreover, the isotropic realizability in the torus needs an extra assumption satisfied by the gradient flow. Similarly, a periodic divergence free field j is said to be isotropically realizable if there exists a positive conductivity σ such that $\sigma^{-1}j$ is a gradient. In [4] we have proved that in dimension three any periodic regular divergence free field is isotropically realizable under some geometrical assumptions. To this end, we have used a more sophistical approach based on three dynamical systems along the current field, its curl and the cross product with its curl. However, the characterization of the current fields is less complete than the characterization of the electric fields.

In this paper we study the isotropic realizability of a strain field for the incompressible Stokes equation in dimension two. More precisely, let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be a regular divergence free field the gradient of which is possibly periodic. The question is the existence of a positive continuous viscosity $\mu : \mathbb{R}^2 \to (0, \infty)$ and a continuous pressure $p : \mathbb{R}^2 \to \mathbb{R}$ such that the symmetrized gradient $e(U) = \frac{1}{2}\big(DU + DU^T\big)$ is solution of the Stokes equation

$$-\operatorname{Div}\big(\mu\, e(U)\big) + \nabla p = 0 \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Since the strain field is a matrix, the dynamical system approach of [5, 4] does not apply. Moreover, we have not succeeded to obtain a global realizability result in $\mathbb{R}^2$ as general as for the electric fields and the current fields in [5, 4]. The difficulty comes from the approach based on the existence of solutions to specific hyperbolic equations.

In Section 2 we study the local realizability of a regular strain field. We prove (see Theorem 2.1) that if the strain field does not vanish at some point of $\mathbb{R}^2$, then the isotropic realizability holds in a neighborhood of that point. Using in equation (1.2) the representation in dimension two of a divergence free gradient field as an orthogonal gradient, we are led to a hyperbolic system which allows us to construct both a suitable viscosity μ and a pressure p.

In Section 3 the question of the global realizability is investigated. To this end we consider the equivalent form of equation (1.2)

$$\operatorname{curl}\big[\operatorname{Div}\big(\mu\, e(U)\big)\big] = 0 \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

for which we search a positive solution of the type $\mu = e^u$. We are thus led to the semilinear wave equation

$$e^{-u}\operatorname{curl}\big[\operatorname{Div}\big(e^u\, e(U)\big)\big] = 0 \quad \text{in } \mathbb{R}^2, \tag{1.4}$$

which must be satisfied by some regular function u in $\mathbb{R}^2$, the coefficients of which depend on the derivatives of the prescribed velocity U. It is well known (see, *e.g.*, [10] and the references therein) that such a nonlinear wave equation does not admit necessarily a global regular solution for given initial data. However, it is not clear that one cannot choose some suitable initial data which induce a global solution. This is the crucial point related to the question of the global realizability. Due to this difficulty we have not obtained a global realizability result but only a quasi-global realizability result under a small perturbation condition. More explicitly, we have proved the following result (see Theorem 3.1): Let M be a matrix in $\mathbb{R}^{2\times 2}$ with zero trace and $M + M^T \neq 0$, and let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be a regular divergence free field such that DU is periodic with average M. For any $R > 0$, if the norm of DU in $C^4(\mathbb{R}^2)^{2\times 2}$ is less than some value $\varepsilon_R > 0$, then the strain field $e(U)$ is isotropically realizable in the disk $D(0, R)$ centered at the origin and of radius R. Unhappily, it is difficult to estimate the value ε_R with respect to R, since it is linked to the lifespan of the solutions u to the semilinear wave equation (1.4), which is not known as above mentioned. So, ε_R could tend to 0 as $R \to \infty$, which would prevent the global isotropic realizability of the strain $e(U)$ in $\mathbb{R}^2$. When the realizability of the strain field is not realizable in the torus, the following alternative holds (see Proposition 3.6):

1. the semilinear wave equation (1.4) has not a global regular solution u,

2. any global regular solution u to (1.4) is either not bounded or not uniformly continuous in $\mathbb{R}^2$.

Section 4 is devoted to singular cases. Section 4.1 deals with the strain field $e(U_\varepsilon)$, $\varepsilon > 0$, defined by

$$e(U_\varepsilon) := \begin{pmatrix} 1 & \varepsilon \sin(2\pi y) \\ \varepsilon \sin(2\pi y) & -1 \end{pmatrix} \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{1.5}$$

which is shown to be not isotropically realizable in the torus (see Proposition 4.1). The main result of Section 3 thus implies that $e(U_\varepsilon)$ is isotropically realizable in the given disk $D(0,R)$ provided that ε is small enough. The strain field $e(U_\varepsilon)$ is actually isotropically realizable in $\mathbb{R}^2$ with the viscosity $\mu(x,y) := e^{2\pi x}$. Hence, by virtue of the above alternative the regular solutions u (including the particular solution $u(x,y) := 2\pi x$) of the equation

$$\operatorname{curl}\left[\operatorname{Div}\big(e^u\, e(U_\varepsilon)\big)\right] = 0 \quad \text{in } \mathbb{R}^2, \tag{1.6}$$

are either not bounded or not uniformly continuous in $\mathbb{R}^2$.

Next, Section 4.2 is devoted to a case where the strain field vanishes at some point. In the electric field framework the Hartman-Wintner theorem (see [8], [12] Chap. 7) claims that if ∇u is a realizable (namely u is solution to some equation (1.1)) regular non-zero electric field in $\mathbb{R}^2$ with $\nabla u(X^*) = 0$, then the critical point X^* is isolated and the following condition holds (see [1], Remark 1.2):

$$\exists\, n \in \mathbb{N}\setminus\{0\},\ \exists\, C \geq 1, \quad C^{-1}\,|X - X^*|^n \leq |\nabla u(X)| \leq C\,|X - X^*|^n \quad \text{for } X \text{ close to } X^*. \tag{1.7}$$

Up to our knowledge there is no similar result for a strain field in $\mathbb{R}^2$ which vanishes at some point. We have simply proved (see Proposition 4.2) that the particular regular strain field only vanishing at the origin:

$$e(U) := \begin{pmatrix} 0 & f(x)+g(y) \\ f(x)+g(y) & 0 \end{pmatrix} \quad \text{where } f,g \in C^\infty\big([-1,1]^2\big) \text{ and } f(0) = g(0) = 0, \tag{1.8}$$

is isotropically realizable in the neighborhood of the origin, if and only if

$$\exists\, a > 0, \quad e(U)(X) = a\,|X|^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o\big(|X|^2\big), \tag{1.9}$$

which is sharper than the Hartman-Wintner condition (1.7).

Finally, in Section 4.3 we study a rank-one laminate strain field which takes only two values and is thus not continuous. We give (see Theorem 4.8) a necessary and sufficient condition on the two phases so that the rank-one laminate strain field is isotropically realizable with a similar rank-one laminate viscosity.

Notations

- I_2 denotes the unit matrix of $\mathbb{R}^2$, and $R_\perp := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- For $A \in \mathbb{R}^{2\times 2}$, A^T denotes the transposed of the matrix A.
- $\cdot$ denotes the scalar product in $\mathbb{R}^d$.
- For any matrices $A, B \in \mathbb{R}^{2\times 2}$, $A : B := \mathrm{tr}\left(A^T B\right)$, where tr denotes the trace of a matrix, and $|A| := \sqrt{\mathrm{tr}\left(A^T A\right)}$ is the Frobenius norm in $\mathbb{R}^{2\times 2}$.
- $\mathbb{R}^{2\times 2}_{s,0}$ denotes the set of the symmetric matrices of $\mathbb{R}^{2\times 2}$ with zero trace.
- For $\xi, \eta \in \mathbb{R}^2$, $\xi \otimes \eta := \left[\xi_i\, \eta_j\right]_{1\leq i,j\leq 2}$ and $\xi \odot \eta := \dfrac{1}{2}\,(\xi\otimes\eta + \eta\otimes\xi)$.
- For $u \in C^1(\mathbb{R}^2)$, the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are respectively denoted $\partial_x u$ and $\partial_y u$. Moreover, the gradient of u is denoted $\nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$.
- For $u \in C^2(\mathbb{R}^2)$, the Hessian matrix of u is denoted $\nabla^2 u := \begin{pmatrix} \partial^2_{xx} u & \partial^2_{xy} u \\ \partial^2_{yx} u & \partial^2_{yy} u \end{pmatrix}$.
- For $U = \begin{pmatrix} U_x \\ U_y \end{pmatrix} \in C^1(\mathbb{R}^2)^2$, the curl of U is $\mathrm{curl}\, U := \partial_x U_y - \partial_y U_x$, the gradient of U is $DU := \begin{pmatrix} \partial_x U_x & \partial_x U_y \\ \partial_y U_x & \partial_y U_y \end{pmatrix}$, and the strain tensor $e(U)$ is defined by

$$e(U) := \frac{1}{2}\left(DU + DU^T\right) = \begin{pmatrix} \partial_x U_x & \frac{1}{2}\,(\partial_x U_y + \partial_y U_x) \\ \frac{1}{2}\,(\partial_x U_y + \partial_y U_x) & \partial_y U_y \end{pmatrix}. \tag{1.10}$$

- For $\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix} \in C^1(\mathbb{R}^d)^{2\times 2}$, the divergence of Σ is $\mathrm{Div}\,(\Sigma) := \begin{pmatrix} \partial_x \Sigma_{xx} + \partial_y \Sigma_{yx} \\ \partial_x \Sigma_{xy} + \partial_y \Sigma_{yy} \end{pmatrix}$.

2 Local realizability for a non-vanishing field

Let $U : \mathbb{R}^2 \to \mathbb{R}^2$, $U = (U_x, U_y)$, be a regular divergence free field, and let $X_* \in \mathbb{R}^2$. The question is to know if the strain field $e(U)$ is *isotropically realizable* for the Stokes equation in the neighborhood of the point X_*. More precisely, does there exist a neighborhood Ω of X_*, a positive continuous viscosity μ in Ω and a continuous pressure p in Ω such that

$$-\,\mathrm{Div}\left(\mu\, e(U)\right) + \nabla p = 0 \quad \text{in } \Omega? \tag{2.1}$$

The following result provides a sufficient condition of local realizability.

Theorem 2.1. *Let U be a divergence free field in $C^4(\mathbb{R}^2)^2$, and let $X_* \in \mathbb{R}^2$ be such that*

$$e(U)(X_*) \neq 0. \tag{2.2}$$

Then, there exist an open neighborhood Ω of X_*, a positive function μ in $C^0(\Omega)$ and a function p in $C^0(\Omega)$, such that the vector field U solves the Stokes equation (2.1) in Ω.

Proof of Theorem 2.1. By a translation we are led to $X_* = (0,0)$. Let Ω be an open disk of positive radius centered on $(0,0)$. Since U is divergence free, it can be written

$$U = R_\perp \nabla u = \begin{pmatrix} -\partial_y u \\ \partial_x u \end{pmatrix}, \quad \text{where } u \in C^5(\bar\Omega). \tag{2.3}$$

Due to the condition (2.2) combined with the regularity of U, we may choose Ω such that

$$\partial^2_{xy} u \neq 0 \ \text{ in } \bar\Omega \quad \text{or} \quad \partial^2_{xx} u - \partial^2_{yy} u \neq 0 \ \text{ in } \bar\Omega. \tag{2.4}$$

Moreover, the change of variables $u'(x', y') = u(x' + y', x' - y')$ yields

$$\partial^2_{xy} u = \frac{1}{2}\left(\partial^2_{x'x'} u' - \partial^2_{y'y'} u'\right). \tag{2.5}$$

Hence, the first condition of (2.4) leads us to the second one. Therefore, from now one we assume that (up to change u in $-u$)

$$\partial^2_{xx} u - \partial^2_{yy} u > 0 \quad \text{in } \bar\Omega. \tag{2.6}$$

The proof is now divided in three steps. In the first step, from two suitable solutions v^+, v^- of second-order hyperbolic equations we build a continuous viscosity μ in the neighborhood of the point $(0,0)$. In the second step, we prove the existence of a function v^+ for $x > 0$. The third step is devoted to the existence of a function v^- for $x < 0$.

First step: Construction of an admissible viscosity μ.
Assume for the moment that there exist a number $\tau > 0$ with $[-\tau, \tau]^2 \subset \Omega$, and two functions $v^+ \in C^2\big([0,\tau] \times [-\tau,\tau]\big)$, $v^- \in C^2\big([-\tau, 0] \times [-\tau, \tau]\big)$ satisfying

$$\begin{cases} \partial^2_{xy} v^+ > 0 & \text{in } [0,\tau] \times [-\tau,\tau] \\ \partial^2_{xy} v^- > 0 & \text{in } [-\tau,0] \times [-\tau,\tau] \\ \partial^2_{xy} v^+(0,\cdot) = \partial^2_{xy} v^-(0,\cdot) & \text{in } [-\tau,\tau] \\ \partial^2_{yy} v^+(0,\cdot) = \partial^2_{yy} v^-(0,\cdot) & \text{in } [-\tau,\tau], \end{cases} \tag{2.7}$$

and such that the divergence free functions $V^+ := R_\perp \nabla v^+$ and $V^- := R_\perp \nabla v^-$ satisfy

$$e(U) : e(V^+) = 0 \ \text{ in } [0,\tau] \times [-\tau,\tau] \quad \text{and} \quad e(U) : e(V^-) = 0 \ \text{ in } [-\tau,0] \times [-\tau,\tau]. \tag{2.8}$$

Then, define the function μ by

$$\mu := \begin{cases} -\dfrac{2\,\partial_x V^+_x}{\partial_x U_y + \partial_y U_x} = \dfrac{2\,\partial^2_{xy} v^+}{\partial^2_{xx} u - \partial^2_{yy} u} & \text{in } [0,\tau] \times [-\tau,\tau] \\ -\dfrac{2\,\partial_x V^-_x}{\partial_x U_y + \partial_y U_x} = \dfrac{2\,\partial^2_{xy} v^-}{\partial^2_{xx} u - \partial^2_{yy} u} & \text{in } [-\tau,0) \times [-\tau,\tau], \end{cases} \tag{2.9}$$

which is continuous and positive in $[-\tau,\tau]^2$ by virtue of (2.6) and (2.7). Also define the function p by

$$p := \begin{cases} \mu\,\partial_x U_x - \partial_y V^+_x = -\mu\,\partial^2_{xy} u + \partial^2_{yy} v^+ & \text{in } [0,\tau] \times [-\tau,\tau] \\ \mu\,\partial_x U_x - \partial_y V^-_x = -\mu\,\partial^2_{xy} u + \partial^2_{yy} v^- & \text{in } [-\tau,0) \times [-\tau,\tau], \end{cases} \tag{2.10}$$

which is continuous in $[-\tau,\tau]^2$ by the fourth condition of (2.7).

By the free divergence of U, (2.8) and the definition (2.9) of μ, we have

$$
\begin{aligned}
p-\mu\,\partial_y U_y-\partial_x V_y^{\pm} &= \mu\,\partial_x U_x-\partial_y V_x^{\pm}-\mu\,\partial_y U_y-\partial_x V_y^{\pm}\\
&= -\,\frac{4\,\partial_x U_x\,\partial_x V_x^{\pm}}{\partial_x U_y+\partial_y U_x}-\partial_y V_x^{\pm}-\partial_x V_y^{\pm}\\
&= -\,\frac{2\,e(U):e(V^{\pm})}{\partial_x U_y+\partial_y U_x}=0.
\end{aligned}
\tag{2.11}
$$

Hence, from (2.9), (2.10) and (2.11) we deduce that U and V are solutions of the system

$$
\begin{cases}
-\,\mu\,\partial_x U_x+p &= -\,\partial_y V_x^{\pm}\\
-\,\dfrac{\mu}{2}\,(\partial_x U_y+\partial_y U_x) &= \partial_x V_x^{\pm}\\
-\,\dfrac{\mu}{2}\,(\partial_x U_y+\partial_y U_x) &= -\,\partial_y V_y^{\pm}\\
-\,\mu\,\partial_y U_y+p &= \partial_x V_y^{\pm}
\end{cases}
\quad \text{in } \big([0,\tau]\times[-\tau,\tau]\big)\cup\big([-\tau,0]\times[-\tau,\tau]\big),
\tag{2.12}
$$

which is equivalent to

$$
-\,\mu\,e(U)+p\,I_2=R_\perp DV^{\pm}\quad \text{in } \big([0,\tau]\times[-\tau,\tau]\big)\cup\big([-\tau,0]\times[-\tau,\tau]\big).
\tag{2.13}
$$

Therefore, we get that

$$
-\operatorname{Div}\big(\mu\,e(U)\big)+\nabla p=\operatorname{Div}\big(R_\perp DV^{\pm}\big)=0\quad \text{in } \big((0,\tau)\times(-\tau,\tau)\big)\cup\big((-\tau,0)\times(-\tau,\tau)\big).
\tag{2.14}
$$

This combined with the continuity of the strain tensor $\mu\,e(U)$ and the pressure p at the interface $\{0\}\times[-\tau,\tau]$ implies that U is solution of the Stokes equation (2.1) replacing Ω by $(-\tau,\tau)^2$.

Second step: Existence of a function v^+ for $x\geq 0$.
Recall that Ω is a regular simply connected neighborhood of $(0,0)$. Let a be the function defined by

$$
a:=-\,\frac{2\,\partial_x U_x}{\partial_x U_y+\partial_y U_x}=\frac{2\,\partial^2_{xy}u}{\partial^2_{xx}u-\partial^2_{yy}u}\quad \text{in } \bar\Omega,
\tag{2.15}
$$

and let A be the matrix-valued function defined by

$$
A:=\begin{pmatrix}\alpha & 0\\ \gamma & \beta\end{pmatrix}\quad \text{in } \Omega,\quad \text{where}\quad
\begin{cases}
\alpha:=a-\sqrt{a^2+1}\\
\beta:=a+\sqrt{a^2+1}\\
\gamma:=-\,\partial_x\alpha-\beta\,\partial_y\alpha.
\end{cases}
\tag{2.16}
$$

Consider the semilinear hyperbolic system given for $V=\begin{pmatrix}v\\ w\end{pmatrix}$ by

$$
\partial_x V+A\,\partial_y V=\begin{pmatrix}\partial_x v+\alpha\,\partial_y v\\ \partial_x w+\beta\,\partial_y w+\gamma\,\partial_y v\end{pmatrix}=\begin{pmatrix}w\\ 0\end{pmatrix}.
\tag{2.17}
$$

System (2.17) is strictly hyperbolic since the eigenvalues of A satisfy $\alpha<\beta$. As u is in $C^5(\bar\Omega)$, the matrix-valued function A belongs to $C^2(\bar\Omega)^{2\times 2}$. Moreover, we can extend the function a in $\mathbb{R}^2\setminus\bar\Omega$ to a function in $C^3_b(\mathbb{R}^2)$ (*i.e.* all the derivatives until order 3 of the function a are bounded in $\mathbb{R}^2$), still denoted by a. Similarly, A can be extended to a function in $C_b^2(\mathbb{R}^2)^{2\times 2}$, still denoted by A. Let $c > 0$ be a constant such that

$$|\alpha| + |\beta| \le c \quad \text{in } \mathbb{R}^2, \tag{2.18}$$

As a consequence, the characteristics associated with system (2.17), $Y(t; x, y)$ and $Z(t; x, y)$ solutions of the ordinary differential equations

$$\begin{cases} \dfrac{\partial Y}{\partial t}(t; x, y) = \alpha\big(t, Y(t; x, y)\big) \\ Y(x; x, y) = y, \end{cases} \quad \begin{cases} \dfrac{\partial Z}{\partial t}(t; x, y) = \beta\big(t, Y(t; x, y)\big) \\ Z(x; x, y) = y, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{2.19}$$

define two functions in $C^3(\mathbb{R}^3)$ (see, *e.g.*, [9] Chapter 17). Let $\mathscr{D}_c$ be the domain defined by

$$\mathscr{D}_c := \big\{(x, y) \in \mathbb{R}^2 \,:\, x \ge 0,\ -1 + cx \le y \le 1 - cx\big\}, \tag{2.20}$$

and let v_0, w_0 be two prescribed functions in $C^2\big([-1, 1]^2\big)$. Then, by the Theorems 3.1 and 3.6 of [3] there exists a unique solution V in $C^2(\mathscr{D}_c)^2$, defined along the characteristics, of the hyperbolic system (2.17) with the initial condition

$$V(0, y) = \begin{pmatrix} v_0(y) \\ w_0(y) \end{pmatrix} \quad \text{for } y \in [-1, 1]. \tag{2.21}$$

Since by (2.17) and (2.19)

$$\frac{d}{dt}\left[v\big(t, Y(t; x, y)\big)\right] = w\big(t, Z(t; x, y)\big), \quad \frac{d}{dt}\left[w\big(t, Z(t; x, y)\big)\right] = -\,(\gamma\,\partial_y v)\big(t, Z(t; x, y)\big), \tag{2.22}$$

we get, taking into account (2.21) and choosing $t = x$, the following integral representation of the solution V,

$$\begin{cases} v(x, y) = v_0\big(Y(0; x, y)\big) + \displaystyle\int_0^x w\big(s, Y(s; x, y)\big)\, ds \\ w(x, y) = w_0\big(Z(0; x, y)\big) - \displaystyle\int_0^x (\gamma\,\partial_y v)\big(s, Z(s; x, y)\big)\, ds, \end{cases} \quad \text{for any } (x, y) \in \mathscr{D}_c. \tag{2.23}$$

Moreover, we have

$$\begin{cases} \dfrac{\partial(\partial_x Y)}{\partial t}(t; x, y) = \partial_y \alpha\big(t, Y(t; x, y)\big)\, \partial_x Y(t; x, y) \\ \alpha(x, y) + \partial_x Y(x; x, y) = \dfrac{\partial}{\partial x}\big(Y(x; x, y)\big) = 0, \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{2.24}$$

and similarly

$$\begin{cases} \dfrac{\partial(\partial_y Y)}{\partial t}(t; x, y) = \partial_y \alpha\big(t, Y(t; x, y)\big)\, \partial_y Y(t; x, y) \\ \partial_y Y(x; x, y) = 1, \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{2.25}$$

Hence, it follows that

$$\begin{cases} \partial_x Y(t; x, y) = -\,\alpha(x, y)\, \exp\left(\displaystyle\int_x^t \partial_y \alpha\big(s; Y(s; x, y)\big)\, ds\right) \\ \partial_y Y(t; x, y) = \exp\left(\displaystyle\int_x^t \partial_y \alpha\big(s; Y(s; x, y)\big)\, ds\right). \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{2.26}$$

On the other hand, consider any constant $\tau > 0$ such that

$$[0,\tau] \times [-\tau,\tau] \subset \mathscr{D}_c. \tag{2.27}$$

Also denote the function v by v^+. Then, (2.23) combined with (2.26) yields

$$\begin{cases} \partial_x v^+(0,y) = w_0(y) - \alpha(0,y)\, v_0'(y) \\ \partial_y v^+(0,y) = v_0'(y), \end{cases} \quad \text{for } y \in [-\tau,\tau]. \tag{2.28}$$

Moreover, the hyperbolic system (2.17) combined with (2.16) implies that

$$\begin{aligned} 0 &= \partial_x \left(\partial_x v^+ + \alpha\, \partial_y v^+\right) + \beta\, \partial_y \left(\partial_x v^+ + \alpha\, \partial_y v^+\right) + \gamma\, \partial_y v^+ \\ &= \partial^2_{xx} v^+ + \alpha\beta\, \partial^2_{yy} v^+ + (\alpha+\beta)\, \partial^2_{xy} v^+, \end{aligned} \tag{2.29}$$

which yields the equation

$$\partial^2_{xx} v^+ - \partial^2_{yy} v^+ + 2a\, \partial^2_{xy} v^+ = 0 \quad \text{in } [0,\tau] \times [-\tau,\tau]. \tag{2.30}$$

Finally, defining the function $V^+ := R_\perp \nabla v^+$ and using the definition of a in (2.15), we deduce from (2.30) that

$$\begin{aligned} e(U) : e(V^+) &= 2\, \partial_x U_x\, \partial_x V_x^+ + \frac{1}{2}\left(\partial_x U_y + \partial_y U_x\right)\left(\partial_x V_y^+ + \partial_y V_x^+\right) \\ &= \frac{1}{2}\left(\partial_x U_y + \partial_y U_x\right)\left(-\, 2a\, \partial_x V_x^+ + \partial_x V_y^+ + \partial_y V_x^+\right) \\ &= \frac{1}{2}\left(\partial_x U_y + \partial_y U_x\right)\left(2a\, \partial^2_{xy} v^+ + \partial^2_{xx} v^+ - \partial^2_{yy} v^+\right) = 0 \ \text{ in } [0,\tau] \times [-\tau,\tau], \end{aligned} \tag{2.31}$$

which corresponds to the first equation of (2.8).

Third step: Existence of a function v^- for $x \le 0$.
Define $\tilde{a}(x,y) := -\, a(-x,y)$ for $x \ge 0$ and $y \in \mathbb{R}$, and let $\tilde{\alpha}$, $\tilde{\beta}$ be the functions defined from $\tilde{a}$ as in formula (2.16). We can also assume that $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the bound (2.18) with some constant $c > 0$. Then, following the approach of the second step, we get that for any $\tilde{v}_0$, $\tilde{w}_0$ in $C^2\big([-1,1]\big)$, there exists a function $\tilde{v} \in C^2(\mathscr{D}_c)$ satisfying for some $\tau > 0$ of (2.27),

$$\begin{cases} \partial_x \tilde{v}(0,y) = \tilde{w}_0(y) - \tilde{\alpha}(0,y)\, \tilde{v}_0'(y) \\ \partial_y \tilde{v}(0,y) = \tilde{v}_0'(y), \end{cases} \quad \text{for } y \in [-\tau,\tau], \tag{2.32}$$

and

$$\partial^2_{xx} \tilde{v} - \partial^2_{yy} \tilde{v} + 2\tilde{a}\, \partial^2_{xy} \tilde{v} = 0 \quad \text{in } [0,\tau] \times [-\tau,\tau]. \tag{2.33}$$

Defining $v^-(x,y) := \tilde{v}(-x,y)$, for $(x,y) \in [-\tau,0] \times [-\tau,\tau]$, we thus deduce from (2.32) and (2.33) that

$$\begin{cases} \partial_x v^-(0,y) = -\, \tilde{w}_0(y) + \tilde{\alpha}(0,y)\, \tilde{v}_0'(y) \\ \partial_y v^-(0,y) = \tilde{v}_0'(y), \end{cases} \quad \text{for } y \in [-\tau,\tau], \tag{2.34}$$

and

$$\partial^2_{xx} v^- - \partial^2_{yy} v^- + 2a\, \partial^2_{xy} v^- = 0 \quad \text{in } [-\tau,0] \times [-\tau,\tau]. \tag{2.35}$$

Moreover, as for (2.30) the equation (2.35) is equivalent to the second equation of (2.8) with $V^- := R_\perp \nabla v^-$.

Finally, choose $v_0(y) = \tilde{v}_0(y) = 0$, $w_0(y) = y$ and $\tilde{w}_0(y) = -y$, for $y \in [-\tau, \tau]$. On the one hand, by (2.28) and (2.34) we have

$$\partial^2_{xy} v^+(0, y) = \partial^2_{xy} v^-(0, y) = 1, \quad \partial^2_{yy} v^+(0, y) = \partial^2_{yy} v^-(0, y) = 0, \quad \text{for } y \in [-\tau, \tau], \tag{2.36}$$

so that the two last equations of (2.7) are satisfied. On the other hand, since

$$\partial^2_{xy} v^+(0, 0) = \partial^2_{xy} v^-(0, 0) = 1,$$

we can take $\tau > 0$ small enough such that the two inequalities of (2.7) also hold. Therefore, the proof of Theorem 2.1 is complete. $\square$

3 Global realizability under a small perturbation

3.1 The main result

Contrary to [4, 5] for fields in Electrostatics, we have not succeeded for the moment to prove a realizability result for strain fields in the whole space or in the torus. The difficulty is strongly linked to the derivation of global regular solutions to nonlinear wave equations. However, we have obtained a nearly global perturbation result for any periodic regular strain field sufficiently close to its average:

Theorem 3.1. *Let M be a matrix in $\mathbb{R}^{2\times 2}$ with zero trace and $M + M^T \neq 0$. Let U be a divergence free function in $C^5(\mathbb{R}^2)^2$ such that $X \mapsto U(X) - MX$ is Y-periodic. For any $R > 0$, there exists $\varepsilon > 0$ only depending on R such that if*

$$\left\| U(X) - MX \right\|_{C^5_\sharp(Y)^{2\times 2}} < \varepsilon, \tag{3.1}$$

then the field $e(U)$ is isotropically realizable in the open disk $D(0, R)$ with a positive viscosity $\mu_R \in C^2\big(D(0, R)\big)$. We can construct an admissible viscosity μ_R so that, for any $S > R > 0$, if (3.1) holds for R and S, then μ_S agrees with μ_R in $D(0, R)$. Moreover, if ε is bounded from below by a positive constant independent of R, the strain field $e(U)$ is isotropically realizable in the whole space $\mathbb{R}^2$ with a positive viscosity $\mu \in C^2(\mathbb{R}^2)$.

The proof of Theorem (3.1) is based on the following perturbation result communicated by P. Gérard [7], which already explains the limitation of the realizability result:

Lemma 3.2. *Let $H(t, z, \lambda)$ be a polynomial of degree two in $\lambda \in \mathbb{R}^2$,*

$$H(t, z, \lambda) = B(t, z)\lambda \cdot \lambda + V(t, z) \cdot \lambda + h(t, z) \quad \textit{for } (t, z) \in \mathbb{R}^2, \ \lambda \in \mathbb{R}^2,$$

the coefficients of which are in $C^2(\mathbb{R}^2)$ in the variables $(t, z) \in \mathbb{R}^2$. Assume that

$$\Lambda := \int_{\mathbb{R}} \big(\|B(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|V(t, \cdot)\|_{L^\infty(\mathbb{R})} \big)\, dt < \infty. \tag{3.2}$$

Then, there exists a positive constant ε depending on Λ such that, if

$$\int_{\mathbb{R}} \|h(t, \cdot)\|_{L^\infty(\mathbb{R})}\, dt < \varepsilon, \tag{3.3}$$

the semilinear wave equation

$$\begin{cases} \Box w(t, z) = \partial^2_{tt} w(t, z) - \partial^2_{zz} w(t, z) = H\big(t, z, \nabla w(t, z)\big), & (t, z) \in \mathbb{R}^2 \\ w(0, z) = \partial_t w(0, z) = 0, & z \in \mathbb{R}, \end{cases} \tag{3.4}$$

has a unique global solution w in $C^1(\mathbb{R}^2)$.

Remark 3.3. The proof of the existence is an adaptation of the energy integral method that can be found for instance in Hörmander's book [10], Chap. VI. Moreover, the uniqueness follows from Theorem 6.4.10 of [10]. On the other hand, the smallness condition (3.3) is essential to control the lifespan of the solution in the Gronwall type estimates. Indeed, P. Gérard has also provided in [7] an example of equation (3.4) whose coefficients have compact support (which thus implies condition (3.2)), and which leads to a blow-up in finite time when the coefficient h is not small enough.

Proof of Theorem 3.1. Let M be a matrix in $\mathbb{R}^{2\times 2}$ with zero trace and $M+M^T\neq 0$. Let U be a divergence free function in $C^5(\mathbb{R}^2)^2$ such that $X\mapsto U(X)-MX$ is Y-periodic. Eliminating the pressure term in (2.1) the strain field $e(U)$ is isotropically realizable in a simply connected domain Ω of $\mathbb{R}^2$ with a positive viscosity $\mu\in C^2(\Omega)$, if and only if

$$\operatorname{curl}\big[\operatorname{Div}\big(\mu\, e(U)\big)\big]=0\quad\text{in }\Omega. \tag{3.5}$$

The natural idea is to search a suitable positive viscosity of type $\mu=e^u$. Then, a lengthy but easy computation shows that μ satisfies the Stokes equation (3.5), if and only if u is solution of the semilinear wave equation

$$A:\nabla^2 u=-A\nabla u\cdot\nabla u+R_\perp\Delta U\cdot\nabla u-\frac{1}{2}\,\Delta(\operatorname{curl}U)\quad\text{in }\Omega, \tag{3.6}$$

where

$$A:=e(U)R_\perp=\begin{pmatrix}\frac{1}{2}\left(\partial_x U_y+\partial_y U_x\right) & -\partial_x U_x\\ -\partial_x U_x & -\frac{1}{2}\left(\partial_x U_y+\partial_y U_x\right)\end{pmatrix}=\begin{pmatrix}a & b\\ b & -a\end{pmatrix}. \tag{3.7}$$

The proof is now divided in three steps. In the first step we globally transform the wave equation (3.6) into a canonical form. In the second step we truncate some of the coefficients in the modified equation (3.6) by coefficients with compact support in view of applying Lemma 3.2. In the third step we use Lemma 3.2 to conclude.

First step: Global transformation into a canonical form.
Due to $M+M^T\neq 0$, up to make the linear change of variables $(x,y)\mapsto(x+y,x-y)$ which permits to commute the entries a and b of the matrix-valued function A (3.7), we can assume that $M_{12}+M_{21}\neq 0$, so that for ε small enough, estimate (3.1) holds with

$$\forall\,(x,y)\in\mathbb{R}^2,\quad |a(x,y)|>\frac{1}{2}\,|M_{12}+M_{21}|-\varepsilon>0. \tag{3.8}$$

The local transformation of the wave equation (3.6) into a canonical form is classical (see, *e.g.*, [6], Section 7.2). The global transformation is perhaps less classical and is based on the following change of variables.

Lemma 3.4. *Consider a matrix M and a vector-valued function U satisfying the assumptions of Theorem 3.1. Let $R(x,\xi)$ and $S(x,\eta)$, for $\xi,\eta\in\mathbb{R}$, be the characteristics defined by*

$$\begin{cases}\partial_x R(x,\xi)=\alpha\big(x,R(x,\xi)\big), & x\in\mathbb{R},\\ R(0,\xi)=\xi\end{cases}\quad\text{and}\quad\begin{cases}\partial_x S(x,\eta)=\beta\big(x,S(x,\eta)\big), & x\in\mathbb{R},\\ S(0,\eta)=\eta\end{cases} \tag{3.9}$$

where $\alpha\neq\beta$ are defined from the matrix-valued function A of (3.7) by

$$\alpha:=\frac{b-\sqrt{a^2+b^2}}{a}\quad\text{and}\quad\beta:=\frac{b+\sqrt{a^2+b^2}}{a}. \tag{3.10}$$

Then, for ε small enough in (3.1), there exist two functions $\xi,\eta\in C^4(\mathbb{R}^2)$ satisfying

$$\forall\,(x,y)\in\mathbb{R}^2,\quad y=R\big(x,\xi(x,y)\big)=S\big(x,\eta(x,y)\big), \tag{3.11}$$

and such that the mapping $(\xi,\eta):(x,y)\mapsto\big(\xi(x,y),\eta(x,y)\big)$ is a C^4-diffeomorphism on $\mathbb{R}^2$ mapping $(0,0)$ to $(0,0)$.

Using the change of variables

$$w(t,z)=u(x,y)\quad\text{where}\quad\begin{cases} t:=\dfrac{\xi(x,y)-\eta(x,y)}{2} \\ z:=\dfrac{\xi(x,y)+\eta(x,y)}{2}\end{cases}\quad\text{for }(x,y)\in\mathbb{R}^2, \tag{3.12}$$

and the following equalities, due to (3.25) below,

$$\begin{cases} \partial_x\xi(x,y)=-\,\alpha\big(x,R(x,\xi(x,y))\big)\,\partial_y\xi(x,y) \\ \partial_x\eta(x,y)=-\,\beta\big(x,S(x,\eta(x,y))\big)\,\partial_y\eta(x,y)\end{cases}\quad\text{for }(x,y)\in\mathbb{R}^2, \tag{3.13}$$

a lengthy but classical computation leads us to

$$A:\nabla^2u=\left(\frac{a^2+2b^2}{a}\,\partial_y\xi\,\partial_y\eta\right)\Box w+\frac{1}{2}\,A:(\nabla^2\xi-\nabla^2\eta)\,\partial_t w+\frac{1}{2}\,A:(\nabla^2\xi+\nabla^2\eta)\,\partial_z w. \tag{3.14}$$

Second step: Truncation of the coefficients.
To this end we need the following result.

Lemma 3.5. *Let $k\in\mathbb{N}$. Each periodic regular function $f\in C^{k+2}_\sharp(Y)$ can be linearly mapped to a function $\varphi_f\in C^k_c(\mathbb{R}^2)$ with compact support such that*

$$\forall\,(x,y)\in\mathbb{R}^2,\quad f(x,y)=\sum_{(p,q)\in\mathbb{Z}^2}\varphi_f(x+p,y+q). \tag{3.15}$$

Moreover, there is a constant $C_k>0$ only depending on k such that

$$\forall\,f\in C^{k+2}_\sharp(Y),\quad \|\varphi_f\|_{C^k(\mathbb{R}^2)}\le C_k\,\|f\|_{C^{k+2}_\sharp(Y)}. \tag{3.16}$$

Using the notation of Lemma 3.5, for any integer $n\in\mathbb{N}$ and any function $f\in C^0_\sharp(Y)$, denote by $[f]_n$ the function defined by the truncation deduced from (3.15),

$$[f]_n(x,y):=\sum_{p,q=-n}^{n}\varphi_f(x+p,y+q)\quad\text{for }(x,y)\in\mathbb{R}^2, \tag{3.17}$$

so that $[f]_0=\varphi_f$ and $[f]_\infty=f$.

Now, return to the nonlinear wave equation (3.6). Due to the compact supports of the functions φ_A, $\varphi_{\Delta U}$, $\varphi_{\Delta(\operatorname{curl}U)}$, for any $R>0$ there exists a smallest integer $n_R\in\mathbb{N}$ such that

$$A=[A]_{n_R},\quad \Delta U=[\Delta U]_{n_R},\quad \Delta(\operatorname{curl}U)=[\Delta(\operatorname{curl}U)]_{n_R}\quad\text{in }D(0,R). \tag{3.18}$$

Then, replacing the functions φ_A, $\varphi_{\Delta U}$, $\varphi_{\Delta(\operatorname{curl}U)}$ in the right-hand side of (3.6) by their truncations at the size $n\in\mathbb{N}$, we get the semilinear wave equation

$$A:\nabla^2u_n=-[A]_n\nabla u_n\cdot\nabla u_n+R_\perp[\Delta U]_n\cdot\nabla u_n-\frac{1}{2}\,[\Delta(\operatorname{curl}U)]_n\quad\text{in }\mathbb{R}^2. \tag{3.19}$$

Next, using the change of variables (3.12) and making a truncation of the coefficients of (3.14) we obtain the modified equation satisfied by the function $w_n(t,z)=u_n(x,y)$,

$$\begin{aligned}&\left(\frac{a^2+2b^2}{a}\,\partial_y\xi\,\partial_y\eta\right)\Box w_n+\frac{1}{2}\,[A]_n:(\nabla^2\xi-\nabla^2\eta)\,\partial_t w_n+\frac{1}{2}\,[A]_n:(\nabla^2\xi+\nabla^2\eta)\,\partial_z w_n\\ &=-[A]_n\nabla u_n\cdot\nabla u_n+R_\perp[\Delta U]_n\cdot\nabla u_n-\frac{1}{2}\,[\Delta(\operatorname{curl}U)]_n\quad\text{in }\mathbb{R}^2\end{aligned} \tag{3.20}$$

Multiplying equation (3.20) by the factor $a(a^2+2b^2)^{-1}\,\partial_\xi R\,\partial_\eta S$ which does not vanish in $\mathbb{R}^2$ (due to (3.8) and to (3.25) below), we are led to the semilinear wave equation

$$\begin{cases} \Box w_n = B_n \nabla w_n \cdot \nabla w_n + V_n \cdot \nabla w_n + h_n, & \text{in } \mathbb{R}^2 \\ w_n(0,z) = \partial_t w_n(0,z) = 0, & z \in \mathbb{R}. \end{cases} \tag{3.21}$$

The functions B_n, V_n, h_n have coefficients in $C^2_c(\mathbb{R}^2)$ since they can be expressed from (3.20) in terms of the truncated functions $[A]_n, [\Delta U]_n, [\Delta(\operatorname{curl} U)]_n$. For example, we have

$$h_n(t,z) := -\,\frac{a\,\partial_\xi R\big(x,\xi(x,y)\big)\,\partial_\eta S\big(x,\eta(x,y)\big)}{2\,(a^2+2b^2)}\,[\Delta(\operatorname{curl} U)]_n(x,y) \quad \text{for } (x,y)\in\mathbb{R}^2, \tag{3.22}$$

which is in $C^2(\mathbb{R}^2)$ since $a,b,R,S,\xi,\eta \in C^4(\mathbb{R}^2)$ and $U \in C^5(\mathbb{R}^2)^2$. Moreover, since $[\Delta(\operatorname{curl} U)]_n$ has compact support and the mapping $(x,y) \mapsto (t,z)$ defined by (3.11), (3.12) is proper (see estimate (3.26) in the proof of Lemma 3.4 below), the function h_n has also compact support with respect to the new variables (t,z).

Third step: Conclusion thanks to Lemma 3.2.
Let $R>0$ and let $n=n_R$ be the integer such that the equalities (3.18) hold. Since B_n, V_n, h_n have coefficients in $C^2_c(\mathbb{R}^2)$, the condition (3.2) is fulfilled with B_n and V_n. By the estimate (3.16) and the definition (3.17) there exists a constant $C>0$ such that

$$\big\|[\Delta(\operatorname{curl} U)]_n\big\|_{C^0(\mathbb{R}^2)} \le C\,\big\|\Delta(\operatorname{curl} U)\big\|_{C^2_\sharp(Y)} \le 4\,C\,\big\|U(X)-MX\big\|_{C^5_\sharp(Y)^{2\times 2}}.$$

This combined with estimate (3.1) and the definition (3.22) of $h_n \in C^2_c(\mathbb{R}^2)$, implies the existence of a constant $C_n>0$ such that

$$\int_{\mathbb{R}} \|h_n(t,.)\|_{L^\infty(\mathbb{R})}\,dt < C_n\,\varepsilon.$$

Since by construction the functions B_n, V_n have compact support, we also have

$$\Lambda_n := \int_{\mathbb{R}} \big(\|B_n(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|V_n(t,\cdot)\|_{L^\infty(\mathbb{R})}\big)\,dt < \infty.$$

Then, by virtue of Lemma 3.2, choosing $\varepsilon=\varepsilon_R>0$ small enough, there exists a global solution $w_n = w_{n_R} \in C^2(\mathbb{R}^2)$ to the equation (3.21), or equivalently, a solution $u_n = u_{n_R} \in C^2(\mathbb{R}^2)$ to the equation (3.19). Finally, using the equalities (3.18) the function u_{n_R} is a solution of the initial wave equation (3.6) in the disk $D(0,R)$, and the function $\mu_R := e^{u_{n_R}}$ solves the equation (3.5) in $\Omega = D(0,R)$. Therefore, the field $e(U)$ is isotropically realizable in $D(0,R)$ with the positive viscosity μ_R.

It remains to prove the global realizability in $\mathbb{R}^2$ under the boundedness condition on ε_R. We have $\mu_R = e^{u_R} \in C^2\big(D(0,R)\big)$, and $u_R(x,y) = w_n(t,z)$ for $(x,y)\in D(0,R)$ (thanks to the change of variables (3.12)), where $w_n = w_{n_R}$ is the unique (see [10], Theorem 6.4.10) solution to the semilinear wave equation (3.21). Hence, if $S>R>0$ and the inequality (3.1) is satisfied with $\varepsilon \le \min(\varepsilon_R,\varepsilon_S)$, then $n_R \le n_S$ and by the uniqueness in equation (3.21) combined with the following equalities (due to (3.18))

$$B_{n_S}(t,z) = B_{n_R}(t,z), \quad V_{n_S}(t,z) = V_{n_R}(t,z), \quad h_{n_S}(t,z) = h_{n_R}(t,z) \qquad \text{for } (x,y)\in D(0,R),$$

we get that $w_{n_R}(t,z) = w_{n_S}(t,z)$ for $(x,y)\in D(0,R)$, so that $\mu_S = \mu_R$ in $D(0,R)$.

Finally, assume that ε_R is bounded from below by a positive constant ε_∞ independent of R. Under the perturbation condition (3.1) with $\varepsilon = \varepsilon_\infty$, we can define the function $\mu \in C^2(\mathbb{R}^2)$ by $\mu := \mu_R$ in $D(0,R)$, for any $R > 0$. The function μ clearly satisfies the equation

$$\operatorname{curl}\left[\operatorname{Div}\left(\mu\, e(U)\right)\right] = 0 \quad \text{in } \mathbb{R}^2.$$

Therefore, the field $e(U)$ is isotropically realizable in the whole space $\mathbb{R}^2$. □

The isotropic realizability in the torus is more intricate since it is connected to the existence of a bounded global solution to the equation (3.6). To illuminate this we have the following result.

Proposition 3.6. *Let $U \in C^3(\mathbb{R}^2)^2$ be a divergence free function such that DU is Y-periodic. Assume that $e(U)$ is not isotropically realizable in the torus. Then, we have the following alternative:*

1. *the semilinear equation* (3.6) *has not a global regular solution,*
2. *any global regular solution to* (3.6) *is either not bounded or not uniformly continuous in $\mathbb{R}^2$.*

Proof. Assume by contradiction that u_0 is a regular solution to equation (3.6), which is bounded and uniformly continuous in $\mathbb{R}^2$. Then, the function $\mu_0 := e^{u_0}$ is uniformly continuous in $\mathbb{R}^2$, and the sequence $(\mu_k)_{k\geq 1}$ defined by

$$\mu_k(x,y) := \frac{1}{(2k+1)^2} \sum_{p,q=-k}^{k} \mu_0(x+p, y+q) \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{3.23}$$

is uniformly bounded and equi-continuous in $\mathbb{R}^2$. Hence, by virtue of Ascoli's theorem μ_k converges uniformly in $\mathbb{R}^2$ to some function $\mu \in C^0(\mathbb{R}^2)$ up to a subsequence. Note that μ is bounded from below in $\mathbb{R}^2$ by a positive constant. Moreover, since we have for any $n \geq 1$,

$$\left|\mu_k(x+1,y) - \mu_k(x,y)\right| + \left|\mu_k(x,y+1) - \mu_k(x,y)\right| \leq \frac{4\,\|\mu_0\|_{L^\infty(\mathbb{R}^2)}}{2k+1} \quad \text{for } (x,y) \in \mathbb{R}^2,$$

it follows that μ is Y-periodic. Next, using that u_0 is solution to equation (3.6) and the Y-periodicity of DU, we get that

$$\operatorname{curl}\left[\operatorname{Div}\left(\mu_k\, e(U)\right)\right] = 0 \quad \text{in } \mathbb{R}^2.$$

Passing to the limit as $k \to \infty$, it follows that at least in the distributions sense

$$\operatorname{curl}\left[\operatorname{Div}\left(\mu\, e(U)\right)\right] = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2), \tag{3.24}$$

which implies that $e(U)$ is isotropically realizable in the torus with the positive periodic continuous function μ. □

Remark 3.7. If we relax the continuity condition on the viscosity in the definition of the isotropic realizability, the alternative of Proposition 3.6 reduces to

1. the semilinear equation (3.6) has not a global regular solution,
2. any global regular solution to (3.6) is not bounded in $\mathbb{R}^2$.

Indeed, in the proof of Proposition 3.6 the sequence of regular functions μ_k now converges weakly-$*$ in $L^\infty(\mathbb{R}^2)$ to some function μ which is not necessarily continuous but still periodic. Therefore, the limit equation (3.24) remains satisfied, which again allows us to conclude.

3.2 Proofs of the technical lemmas

Proof of Lemma 3.4. Let $(x,y)\in\mathbb{R}^2$. Since for any $\xi\in\mathbb{R}$,

$$\partial_\xi R(x,\xi)=\exp\left(\int_0^x \partial_y\alpha\big(t,R(t,\xi)\big)\,dt\right)>0 \quad\text{and}\quad \big|R(x,\xi)-\xi\big|\le\|\alpha\|_{L^\infty(\mathbb{R}^2)}\,|x|,$$

the mapping $\xi\mapsto R(x,\xi)$ is a C^1-diffeomorphism on $\mathbb{R}$. Hence, there exists a unique $\xi(x,y)\in\mathbb{R}$ such that $y=R\big(x,\xi(x,y)\big)$. Again using that $\partial_\xi R\neq 0$ and that $R\in C^4(\mathbb{R}^2)$ (recall that $U\in C^5(\mathbb{R}^2)$), the implicit function theorem implies that $\xi\in C^4(\mathbb{R}^2)$. Similarly, the function η defined implicitly by $y=S\big(x,\eta(x,y)\big)$ belongs to $C^4(\mathbb{R}^2)$. Moreover, by the chain rule applied to (3.11) using (3.9) we get that

$$\begin{cases} 0=\alpha\big(x,R(x,\xi(x,y))\big)+\partial_\xi R\big(x,\xi(x,y)\big)\,\partial_x\xi(x,y)\\ 0=\beta\big(x,R(x,\xi(x,y))\big)+\partial_\eta S\big(x,\eta(x,y)\big)\,\partial_x\eta(x,y)\\ 1=\partial_\xi R\big(x,\xi(x,y)\big)\,\partial_y\xi(x,y)\\ 1=\partial_\eta S\big(x,\eta(x,y)\big)\,\partial_y\eta(x,y), \end{cases}\quad\text{for }(x,y)\in\mathbb{R}^2. \tag{3.25}$$

Hence, the Jacobian $J_{(\xi,\eta)}$ of the mapping (ξ,η) satisfies

$$J_{(\xi,\eta)}=(\beta-\alpha)\,\partial_y\xi\,\partial_y\eta=\frac{\beta-\alpha}{\partial_\xi R\,\partial_\eta S}\neq 0\quad\text{in }\mathbb{R}^2.$$

Let us now prove that the mapping (ξ,η) is proper, *i.e.* the reciprocal of any compact set K in $\mathbb{R}^2$ is compact in $\mathbb{R}^2$. Consider $(x,y)\in\mathbb{R}^2$ such that $\big(\xi(x,y),\eta(x,y)\big)\in K$. By (3.9) and (3.11) we have

$$\xi(x,y)-\eta(x,y)=\int_0^x\big[\partial_t S(t,\eta)-\partial_t R(t,\xi)\big]\,dt=\int_0^x\big[\beta\big(t,S(t,\eta)\big)-\alpha\big(t,R(t,\xi)\big)\big]\,dt.$$

This combined with the definition (3.10) of α,β and the estimates (3.1), (3.8) satisfied by a, b, implies that for ε small enough,

$$\big|\xi(x,y)-\eta(x,y)\big|\ge c_K\,|x|,$$

where $c_K>0$ only depends on K. Moreover, by (3.11) and (3.9) we have

$$\big|y-\xi(x,y)\big|=\big|R\big(x,\xi(x,y)\big)-R\big(0,\xi(x,y)\big)\big|\le C_K\,|x|,$$

where $C_K>0$ only depends on K. The two former estimates yield that

$$|x|+|y|\le\big|\xi(x,y)\big|+\frac{C_K+1}{c_K}\,\big|\xi(x,y)-\eta(x,y)\big|, \tag{3.26}$$

which shows that (x,y) lies in a compact set of $\mathbb{R}^2$. By virtue of Hadamard's theorem the properness of the C^4-mapping (ξ,η) and the non-vanishing of its Jacobian imply that it is a C^4-diffeomorphism on $\mathbb{R}^2$. □

Proof of Lemma 3.5. Let $f\in C^{k+2}_\sharp(Y)$, $k\in\mathbb{N}$. Fix $\theta\in C^\infty_c(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2}\theta(X)\,dX=1$. Define h by the convolution $h:=\theta*1_Y$ (recall that $Y=[0,1]^2$), and the function φ_f by

$$\varphi_f(x,y):=\sum_{(p,q)\in\mathbb{Z}^2}\hat f(p,q)\,e^{2i\pi(px+qy)}\,h(x,y)\quad\text{for }(x,y)\in\mathbb{R}^2, \tag{3.27}$$

where $\hat{f}(p,q)$ denotes the Fourier coefficient of f given by

$$\hat{f}(p,q) := \int_Y f(x,y)\, e^{-2i\pi\,(px+qy)}\, dx\, dy \quad \text{for } (p,q) \in \mathbb{Z}^2.$$

Using that

$$\widehat{\partial_x f}(p,0) = 2i\pi p\hat{f}(p,0), \quad \widehat{\partial_y f}(0,q) = 2i\pi q\hat{f}(0,q), \quad \widehat{\partial^2_{xy} f}(p,q) = -4\pi^2 pq\hat{f}(p,q),$$

Cauchy-Schwarz' inequality and Parseval's identity imply that

$$\sum_{(p,q)\in\mathbb{Z}^2} \big|\hat{f}(p,q)\big| \le \big|\hat{f}(0,0)\big| + \sum_{p\in\mathbb{Z}\setminus\{0\}} \frac{\big|\widehat{\partial_x f}(p,0)\big|}{2\pi|p|} + \sum_{q\in\mathbb{Z}\setminus\{0\}} \frac{\big|\widehat{\partial_y f}(0,q)\big|}{2\pi|q|} + \sum_{(p,q)\in(\mathbb{Z}\setminus\{0\})^2} \frac{\big|\widehat{\partial_{xy} f}(p,q)\big|}{4\pi^2|pq|}$$

$$\le c\,\|f\|_{L^2(Y)} + c\,\|\partial_x f(\cdot,0)\|_{L^2(Y)} + c\,\|\partial_y f(0,\cdot)\|_{L^2(Y)} + c\,\|\partial^2_{xy} f\|_{L^2(Y)} \le C\,\|f\|_{C^2_\sharp(Y)}.$$

Hence, it follows that $\varphi_f \in C^0_c(\mathbb{R}^2)$ and estimate (3.16) is satisfied for $k = 0$. Iterating we get that $\varphi_f \in C^k_c(\mathbb{R}^2)$ and (3.16) holds for any $k \in \mathbb{N}$.

Now, consider the function g defined by

$$g(x,y) := \sum_{(p,q)\in\mathbb{Z}^2} \varphi_f(x+p, y+q) \quad \text{for } (x,y) \in \mathbb{R}^2,$$

which is clearly in $C^k_\sharp(Y)$. We also have $\hat{g}(p,q) = \mathscr{F}(\varphi_f)(p,q)$ for any $(p,q) \in \mathbb{Z}^2$, where $\mathscr{F}$ denotes the Fourier transform in $L^1(\mathbb{R}^2)$. On the other hand, by the definition (3.27) of φ_f we have

$$\begin{aligned}
\mathscr{F}(\varphi_f)(p,q) &= \sum_{(j,k)\in\mathbb{Z}^2} \hat{f}(j,k) \left(\int_{\mathbb{R}^2} e^{2i\pi\,[(j-p)x+(k-q)y]}\, h(x,y)\, dx\, dy \right) \\
&= \sum_{(j,k)\in\mathbb{Z}^2} \hat{f}(j,k)\, \mathscr{F}(h)(p-j, q-k),
\end{aligned}$$

and since $\mathscr{F}(\theta)(0,0) = 1$,

$$\mathscr{F}(h)(p-j,q-k) = \mathscr{F}(\theta)(p-j,q-k)\,\mathscr{F}(1_Y)(p-j,q-k) = \begin{cases} 1 & \text{if } (j,k) = (p,q) \\ 0 & \text{if } (j,k) \neq (p,q). \end{cases}$$

Therefore, we get that $\hat{g}(p,q) = \hat{f}(p,q)$ for any $(p,q) \in \mathbb{Z}^2$, which implies that $g = f$ and thus the representation formula (3.15). $\square$

4 A few singular cases

4.1 An example of non-isotropic realizability in the torus

Let U_ε, $\varepsilon > 0$, be the divergence free vector-valued function defined in $\mathbb{R}^2$ by

$$U_\varepsilon(x,y) := \begin{pmatrix} x - \dfrac{\varepsilon}{\pi} \cos(2\pi y) \\ -\,y \end{pmatrix} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

The associated field

$$e(U_\varepsilon) = \begin{pmatrix} 1 & \varepsilon\sin(2\pi y) \\ \varepsilon\sin(2\pi y) & -1 \end{pmatrix} \quad \text{for } (x,y) \in \mathbb{R}^2, \tag{4.1}$$

is a smooth and Y-periodic perturbation of the constant matrix $\mathrm{diag}\,(1,-1)$. We have the following result of non-realizability:

Proposition 4.1. *The periodic field $e(U_\varepsilon)$ defined by* (4.1) *is isotropically realizable in $\mathbb{R}^2$, but not in the torus. Moreover, the semilinear wave equation* (3.6) *associated with $U = U_\varepsilon$ has a global regular solution, and any global regular solution to* (3.6) *is not bounded or not uniformly continuous in $\mathbb{R}^2$.*

Proof. Assume by contradiction that $e(U_\varepsilon)$ is isotropically realizable with a continuous positive conductivity $\mu(x, y)$ and a continuous pressure $p(x, y)$ which are both 1-periodic with respect to the variable x. Set $Q := (0, 1) \times (-r, r)$, with $r \in (0, 1/2)$. Integrating by parts and using the periodicity with respect to x, we get that (ν denotes the outside normal to ∂Q)

$$\begin{aligned}
0 &= \int_Q \big[\mathrm{Div}\,(\mu\, e(U_\varepsilon)) - \nabla p\big] \cdot e_x \, dx\, dy = \int_{\partial Q} \mu\, e(U_\varepsilon) : (e_x \otimes \nu)\, ds + 0 \\
&= \int_0^1 \big[\big(\mu\, e(U_\varepsilon)\big)(x, r) : (e_x \otimes e_y) - \big(\mu\, e(U_\varepsilon)\big)(x, -r) : (e_x \otimes e_y)\big]\, dx \\
&= \varepsilon \sin(2\pi r) \int_0^1 \big[\mu(x, r) + \mu(x, -r)\big]\, dx > 0.
\end{aligned}$$

This contradiction shows that for any $\varepsilon > 0$, the periodic field $e(U_\varepsilon)$ is not isotropically realizable in the torus as a strain field. On the contrary, the isotropic realizability holds clearly for $\varepsilon = 0$, since $e(U_0) = \mathrm{diag}\,(1, -1)$.

The semilinear wave equation (3.6) associated with U_ε reads as

$$\begin{aligned}
&2\, \partial^2_{xy} u - \varepsilon \sin(2\pi y)\, \partial^2_{xx} u + \varepsilon \sin(2\pi y)\, \partial^2_{yy} u \\
&= -\, 2\, \partial_x u\, \partial_y u + \varepsilon \sin(2\pi y)\, (\partial_x u)^2 - \varepsilon \sin(2\pi y)\, (\partial_y u)^2 \\
&+ 4\pi\varepsilon \cos(2\pi y)\, \partial_y u - 4\pi^2 \varepsilon \sin(2\pi y),
\end{aligned} \tag{4.2}$$

which clearly has $u(x, y) = 2\pi x$ as a global solution in $\mathbb{R}^2$. However, by virtue of Proposition 3.6 we have the following alternative:

1. equation (4.2) has not a global regular solution,
2. any global regular solution to (4.2) is either not bounded or not uniformly continuous in $\mathbb{R}^2$.

Therefore, the second alternative holds for equation (4.2). □

4.2 An example with a vanishing field

The case where the strain field $e(U)$ vanishes at one point X_* is more delicate. For the moment we have no general result. However, the following case with separate variables already shows the difficulties of the problem.

Proposition 4.2. *Let f and g be two functions in $C^0([-1, 1])$ satisfying*

$$f(0) = g(0) = 0 \quad \text{and} \quad \forall\, x \in [-1, 1] \setminus \{0\}, \;\; f(x) > 0, \;\; g(x) > 0, \tag{4.3}$$

and having asymptotic expansions of any order at the point 0.
Consider the strain field $e(U)$ defined by

$$e(U) := \begin{pmatrix} 0 & f(x) + g(y) \\ f(x) + g(y) & 0 \end{pmatrix} \quad \text{for } (x, y) \in [-1, 1]^2. \tag{4.4}$$

Then, a necessary and sufficient condition for $e(U)$ to be isotropically realizable for the incompressible Stokes equation with a continuous function $\mu > 0$ in a neighborhood of $(0,0)$, is that there exists $a > 0$ such that

$$f(x) = a\,x^2 + o(x^2) \quad \text{and} \quad g(x) = a\,x^2 + o(x^2). \tag{4.5}$$

Remark 4.3. The strain field $e(U)$ defined by (4.4) is for example associated with the divergence free field given by

$$U(x,y) = 2\begin{pmatrix} \displaystyle\int_0^y g(t)\,dt \\ \displaystyle\int_0^x f(t)\,dt \end{pmatrix} \quad \text{for } (x,y) \in [-1,1]^2. \tag{4.6}$$

Moreover, due to condition (4.3) the strain field $e(U)$ only vanishes at the point $(0,0)$ in the neighborhood of this point.

Remark 4.4. If we relax the continuity assumption of μ at the point $(0,0)$, then the necessary and sufficient condition of realizability becomes

$$\exists\, n \in 2\mathbb{N},\ \exists\, a, b > 0, \quad f(x) = a\,x^n + o(x^n) \quad \text{and} \quad g(x) = b\,x^n + o(x^n). \tag{4.7}$$

This is induced by the third step of the proof of Proposition 4.2 below.

Proof of Proposition 4.2. The proof is divided in three steps according to the expansions of f, g at the point 0. In the sequel, for any nonnegative functions φ, ψ being continuous in a neighborhood of $(0,0)$, we denote $\varphi \approx \psi$ when there exists a constant $c > 1$ such that $c^{-1}\varphi \le \psi \le c\,\varphi$ in a neighborhood of $(0,0)$.

First case: $\forall\, k \in \mathbb{N}$, $f(x) = o(x^k)$ and $g(x) = o(x^k)$.
Assume that $e(U)$ is realizable with a positive continuous viscosity μ on a non-empty open disk Ω centered on $(0,0)$. Then, there exists a pressure $p \in L^2(\Omega)$ such that the Stokes equation (2.1) holds. Hence, we have

$$\partial_y\big(\mu\,(f(x) + g(y))\big) = \partial_x p \quad \text{and} \quad \partial_x\big(\mu\,(f(x) + g(y))\big) = \partial_y p \quad \text{in } \Omega, \tag{4.8}$$

which implies that

$$\partial^2_{xx}\big(\mu\,(f(x) + g(y))\big) - \partial^2_{yy}\big(\mu\,(f(x) + g(y))\big) = 0 \quad \text{in } \Omega. \tag{4.9}$$

Therefore, there exist two continuous functions F, G defined around the point 0 such that

$$\mu(x,y)\,(f(x) + g(y)) = F(x+y) + G(x-y) \quad \text{in a neighborhood of } (0,0). \tag{4.10}$$

Since μ is positive and continuous in the neighborhood of $(0,0)$, the previous equality yields

$$f(x) + g(y) \approx F(x+y) + G(x-y). \tag{4.11}$$

We have $F(0) + G(0) = 0$ so that we can assume that $F(0) = G(0) = 0$ replacing F and G by $F - F(0)$ and $G - G(0)$. Taking successively $y = 0$, $x = 0$, $y = x$ and $y = -x$ in (4.11), we get that

$$\begin{cases} f(x) \approx F(x) + G(x), & g(x) \approx F(x) + G(-x), \\ F(2x) \approx f(x) + g(x), & G(2x) \approx f(x) + g(-x), \end{cases} \tag{4.12}$$

which implies that

$$f(x) \approx f(x/2) + g(x/2) + g(-x/2) \quad \text{and} \quad g(x) \approx f(x/2) + f(-x/2) + g(x/2). \tag{4.13}$$

Hence, the even nonnegative function h defined by $h(x) := f(x)+f(-x)+g(x)+g(-x)$ satisfies

$$h(x) \approx h(x/2) \quad \text{and} \quad \forall\, k \in \mathbb{N},\ \ h(x) = x^k\,\varepsilon_k(x) \ \text{ with } \lim_{x\to 0} \varepsilon_k(x) = 0. \tag{4.14}$$

Reiterating the first condition of (4.14) there exists $c > 1$ such that

$$\forall\, n, k \in \mathbb{N}, \quad h(x) \le c^n\, h(2^{-n}x) = c^n\, 2^{-kn}\, x^k\, \varepsilon_k(2^{-n}x), \quad \text{for } x > 0 \text{ close to } 0. \tag{4.15}$$

Then, choosing $k \in \mathbb{N}$ such that $2^{-k}c \le 1$, it follows that

$$0 \le h(x) \le \lim_{n\to\infty} (2^{-k}c)^n\, x^k\, \varepsilon_k(2^{-n}x) = 0, \tag{4.16}$$

which yields $h(x) = 0$ and contradicts the assumption (4.3) on f, g. Therefore, the isotropic realizability cannot be satisfied in this case.

Second case: $\exists\, m \in \mathbb{N},\ \exists\, a \neq 0,\ f(x) = a\,x^m + o(x^m)$ and $\forall\, k \in \mathbb{N},\ g(x) = o(x^k)$.
By (4.3) m is an even positive integer and $a > 0$. This combined with (4.12) yields

$$\begin{cases} F(x) + G(-x) \approx g(x), \\ F(x) \approx a\,(x/2)^m + g(x/2) \approx x^m, \quad G(-x) \approx a\,(-x/2)^m + g(x/2) \approx x^m. \end{cases} \tag{4.17}$$

Hence, we obtain that

$$x^m \approx F(x) + G(-x) \approx g(x), \tag{4.18}$$

which leads us to a contradiction. Therefore, the isotropic realizability cannot hold in this case.

Third case: $\exists\, m, n \in \mathbb{N},\ \exists\, a, b \neq 0,\ f(x) = a\,x^m + o(x^m)$ and $g(x) = b\,x^n + o(x^n)$.
By (4.3) m, n are even positive integers and $a, b > 0$. Again by (4.12) we have

$$F(x) + G(x) \approx x^m, \quad F(x) + G(-x) \approx x^n, \quad F(x) \approx x^m + x^n, \quad G(\pm\, x) \approx x^m + x^n. \tag{4.19}$$

Hence, we get that

$$x^m \approx x^m + x^n \approx x^n, \tag{4.20}$$

which implies that $m = n$.

On the other hand, the continuity of μ (4.10) at $(0,0)$ implies that

$$\lim_{(x,y)\to(0,0)} \frac{F(x+y) + G(x-y)}{f(x) + g(y)} = \ell := \mu(0,0) > 0. \tag{4.21}$$

Taking successively $y = x$, $y = -x$, $y = 0$ and $x = 0$ in (4.21), we get that

$$\begin{cases} F(2x) \underset{0}{\sim} \ell\,(a+b)\,x^n, \quad G(2x) \underset{0}{\sim} \ell\,(a+b)\,x^n, \\ F(x) + G(x) \underset{0}{\sim} \ell\,a\,x^n, \quad F(x) + G(-x) \underset{0}{\sim} \ell\,b\,x^n, \end{cases} \tag{4.22}$$

which implies that $2(a+b) = 2^n a = 2^n b$. Therefore, we deduce that $a = b$ and $n = 2$.

Conversely, if the asymptotic expansions (4.5) hold, then the function μ defined by

$$\mu(x,y) := \begin{cases} \dfrac{x^2 + y^2}{f(x) + g(y)} & \text{if } (x,y) \in [-1,1]^2 \setminus \{(0,0)\} \\[2mm] \dfrac{1}{a} & \text{if } (x,y) = (0,0), \end{cases} \tag{4.23}$$

is positive and continuous in $[-1,1]^2$. Moreover, by (4.4) we have

$$\mathrm{Div}\,\big(\mu\, e(U)\big) = \mathrm{Div}\begin{pmatrix} 0 & x^2+y^2 \\ x^2+y^2 & 0 \end{pmatrix} = \nabla\,(2\,xy)\,. \tag{4.24}$$

Therefore, the strain field $e(U)$ is realizable with the continuous positive function μ in a neighborhood of $(0,0)$, which concludes the proof of Proposition 4.2. □

4.3 The case of a laminate field

Theorem 2.1 does not hold under the sole condition (2.2) for non-regular fields. An example of this situation is given by the laminates.

Definition 4.5. A two-phase rank-one laminate field is any measurable function $E : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ defined by

$$E(X) := \chi(\xi\cdot X)\, E_1 + \big(1-\chi(\xi\cdot X)\big)\, E_2 \quad \text{for a.e. } X=(x,y)\in\mathbb{R}^2, \tag{4.25}$$

where E_1, E_2 are two given matrices in $\mathbb{R}^{2\times 2}$, ξ is a given unit norm vector in $\mathbb{R}^2$, and χ is a characteristic function in $L^\infty(\mathbb{R})$.

We have the following characterization of rank-one laminates:

Proposition 4.6. *A two-phase rank-one laminate field E of type* (4.25) *is a strain field $e(U)$ for some divergence free Lipschitz function $U : \mathbb{R}^2 \to \mathbb{R}^2$, if and only if*

$$E_1, E_2 \in \mathbb{R}^{2\times 2}_{s,0} \quad \textit{and} \quad \exists\,\lambda\in\mathbb{R},\;\; E_1 - E_2 = \lambda\,\xi\odot R_\perp\xi. \tag{4.26}$$

Proof. Assume that the field E of (4.25) agrees with $e(U)$ for some divergence free Lipschitz function $U : \mathbb{R}^2 \to \mathbb{R}^2$. Since the strain field $e(U)$ is a zero trace symmetric matrix-valued function, the phases E_1, E_2 belong to $\mathbb{R}^{2\times 2}_{s,0}$ (symmetric with zero trace). Moreover, the strain field $e(U)$ satisfies the differential constraint

$$\partial^2_{xx}\big[e(U)_{22}\big] + \partial^2_{yy}\big[e(U)_{11}\big] = 2\,\partial^2_{xy}\big[e(U)_{12}\big] \quad \text{in } \mathbb{R}^2, \tag{4.27}$$

which by (4.25) implies that

$$\big(\xi_x^2 - \xi_y^2\big)\,(E_1-E_2)_{22} = -\,2\,\xi_x\xi_y\,(E_1-E_2)_{12}. \tag{4.28}$$

Since $\xi_x^2-\xi_y^2$ and $\xi_x\xi_y$ are not simultaneously zero ($|\xi|=1$) and $E_1-E_2\in\mathbb{R}^{2\times 2}_{s,0}$, we deduce from the previous equality the existence of $\lambda\in\mathbb{R}$ such that

$$E_1 - E_2 = \lambda\begin{pmatrix} -2\,\xi_x\xi_y & \xi_x^2-\xi_y^2 \\ \xi_x^2-\xi_y^2 & 2\,\xi_x\xi_y \end{pmatrix} = \lambda\,\xi\odot R_\perp\xi. \tag{4.29}$$

Conversely, assume that (4.26) holds. Consider the Lipschitz function U defined by

$$U(X) := E_2\, X + \lambda\left(\int_0^{\xi\cdot X}\chi(t)\,dt\right) R_\perp\xi \quad \text{for } X\in\mathbb{R}^2. \tag{4.30}$$

We have

$$DU(X) = E_2 + \lambda\,\chi(\xi\cdot X)\,\xi\otimes R_\perp\xi, \quad \text{for a.e. } X\in\mathbb{R}^2, \tag{4.31}$$

hence by (4.26) and (4.25)

$$e(U)(X) = E_2 + \chi(\xi\cdot X)\,(E_1-E_2) = E(X) \quad \text{for a.e. } X\in\mathbb{R}^2, \tag{4.32}$$

which concludes the proof. □

Now, define the isotropic realizability for laminates.

Definition 4.7. A two-phase rank-one laminate field E of type (4.25) is isotropically realizable for the Stokes equation in $\mathbb{R}^2$ if there exist a two-phase rank-one positive function μ defined by

$$\mu(X) := \chi(\xi\cdot X)\,\mu_1 + \big(1-\chi(\xi\cdot X)\big)\,\mu_2 \quad \text{for a.e. } X\in\mathbb{R}^2, \quad \text{with } \mu_1,\mu_2>0, \tag{4.33}$$

and a function $p\in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that in the distributions sense,

$$-\operatorname{Div}(\mu\,E)+\nabla p = 0 \quad \text{in } \mathbb{R}^2. \tag{4.34}$$

Then, we have the following realizability result for two-phase rank-one laminates.

Theorem 4.8. *A strain field E of type* (4.25) *is isotropically realizable in the sense of Definition 4.7, if and only if*

$$E_1 : E_2 > \frac{|E_1|^2|E_2|^2+(E_1:E_2)^2}{|E_1|^2+|E_2|^2} \quad \textit{or} \quad E_1 = E_2. \tag{4.35}$$

Remark 4.9. When $E_1\neq E_2$, the realizability condition (4.35) is stronger than the condition (2.2), *i.e.* $E\neq 0$, of the regular case. It depends only on the values of the strain field in the phases and not on the lamination direction.

Proof of Theorem 4.8. Consider a strain field E of type (4.25) and a positive function μ of type (4.33). We have in the distributions sense

$$\operatorname{Div}(\mu\,E) = \chi'(\xi\cdot X)\,(\mu_1\,E_1-\mu_2\,E_2)\,\xi \quad \text{in } \mathbb{R}^2. \tag{4.36}$$

Hence, if equation (4.34) holds with a function $p\in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, then $\nabla p = \chi'(\xi\cdot X)\,q$ in $\mathbb{R}^2$ for some fixed vector $q\in\mathbb{R}^2$, which implies that $q\parallel\xi$. Thus, p is also a two-phase rank-one laminate function, *i.e.*

$$p(x) = \chi(\xi\cdot X)\,p_1 + \big(1-\chi(\xi\cdot X)\big)\,p_2 \quad \text{for a.e. } X\in\mathbb{R}^2. \tag{4.37}$$

It follows that E solves the Stokes equation (4.34) with μ and p if and only if

$$(\mu_1\,E_1-\mu_2\,E_2)\,\xi = (p_1-p_2)\,\xi. \tag{4.38}$$

Therefore, E is isotropically realizable in the sense of Definition 4.8 if and only if there exist two constants $\mu_1,\mu_2>0$ such that $(\mu_1\,E_1-\mu_2\,E_2)\,\xi\parallel\xi$, or equivalently

$$\exists\,\mu_1,\mu_2>0,\quad \mu_1\,E_1\,R_\perp\xi\cdot\xi = \mu_2\,E_2R_\perp\xi\cdot\xi. \tag{4.39}$$

Next, let us check that condition (4.39) is equivalent to the following one

$$(E_1R_\perp\xi\cdot\xi)\,(E_2R_\perp\xi\cdot\xi)>0 \quad \text{or} \quad E_1=E_2. \tag{4.40}$$

It is clear that condition (4.40) implies (4.39). Conversely, if (4.39) holds then

$$(E_1R_\perp\xi\cdot\xi)\,(E_2R_\perp\xi\cdot\xi)\geq 0. \tag{4.41}$$

To obtain (4.40) it is enough to deal with the case of equality in (4.41). This combined with (4.39) implies that $E_1R_\perp\xi\cdot\xi = E_2R_\perp\xi\cdot\xi = 0$. However, by the jump condition of (4.26) we have

$$E_1R_\perp\xi\cdot\xi - E_2R_\perp\xi\cdot\xi = \frac{\lambda}{2}\left[(\xi\otimes R_\perp\xi)R_\perp\xi\cdot\xi + (R_\perp\xi\otimes\xi)R_\perp\xi\cdot\xi\right] = \frac{\lambda}{2}\,|R_\perp\xi|^2\,|\xi|^2 = \frac{\lambda}{2}, \tag{4.42}$$

which yields $\lambda = 0$. Therefore, again by (4.26) we get the desired equality $E_1 = E_2$.

Finally, noting that $E_i R_\perp \xi \cdot \xi = E_i : (\xi \odot R_\perp \xi)$ for $i = 1, 2$, and using equality (4.26) we obtain that

$$\begin{aligned} \lambda^2 \left(E_1 R_\perp \xi \cdot \xi\right) \left(E_2 R_\perp \xi \cdot \xi\right) &= \big(E_1 : (E_1 - E_2)\big) \big(E_2 : (E_1 - E_2)\big) \\ &= \big(|E_1|^2 + |E_2|^2\big)\, E_1 : E_2 - |E_1|^2 |E_1|^2 - (E_1 : E_2)^2, \end{aligned} \tag{4.43}$$

which implies the equivalence between (4.35) and (4.40). The proof of Theorem 4.8 is now complete. ☐

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